A MARTINGALE INEQUALITY FOR THE SQUARE
AND MAXIMAL FUNCTIONS

BY

LOUIS H.Y. CHEN

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1.   Introduction and Notation.

An inequality involving a class of functions for weak martingales and nonnegative weak submartingales is proved. Three special cases are deduced, one of which generalizes and refines a result of Austin (1966). As an application of the inequality, the special cases are used to give easy proofs of Burkholder's (1966) $L \log L$ and L_p (for $1 < p \leq 2$) inequalities for the square function of a martingale or a nonnegative submartingale. Although the inequality and the special cases are proved for weak martingales and nonnegative weak submartingales, they are also new for martingales and nonnegative submartingales.

Weak martingales and weak submartingales were first defined in Nelson (1970). Examples of these can be found in Nelson (1970) and Berman (1976). For ease of reference, we give the definition here. A sequence $f = (f_1, f_2, \ldots)$ defined on a probability space is a weak martingale (or weak submartingale) if f_n is integrable and $E(f_{n+1} | f_n) = (\text{or} \geq)$ f_n a.s. for $n \geq 1$. Of course, a martingale (or submartingale) is a weak martingale (or weak submartingale).

Throughout this paper, unless otherwise stated, $f = (f_1, f_2, \ldots)$ will denote a weak martingale or a nonnegative weak submartingale. As usual $f_0 \equiv 0$. The difference sequence of f will be denoted by

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$\mathbf{d} = (d_1, d_2, \ldots)$. Also $S_n(f) = (\sum_{i=1}^{n} d_i^2)^{1/2}$, $S(f) = \sup_{1 \leq n < \infty} S_n(f)$,

$f_n^* = \sup_{1 \leq i \leq n} |f_i|$, $f^* = \sup_{1 \leq n < \infty} f_n^*$ and $\|f\|_p = \sup_{1 \leq n < \infty} \|f_n\|_p$ for

$1 \leq p < \infty$. All functions are real-valued, Borel measurable and

defined on the real line.

2. <u>The Main Results.</u>

We first derive an identity.

<u>Lemma 2.1.</u> Let φ be a differentiable function whose derivative φ' is

an indefinite integral of φ'' such that $\varphi(0) = \varphi'(0) = 0$, φ'' is a

nonnegative and even function, and such that for $n \geq 1$, $f_n \varphi'(f_n)$ is

integrable. Define $K_i(x) = (d_i - x)^+$ if $x \geq 0$ and $= (d_i - x)^-$ if

$x < 0$, $i \geq 1$. Then for $n \geq 1$, $\varphi(f_n)$ is integrable, and

$$(2.1) \qquad E\varphi(f_n) \geq \sum_{i=1}^{n} E\int_{-\infty}^{\infty} \varphi''(f_{i-1} + x) K_i(x) dx$$

where equality holds in the weak martingale case. Furthermore for

$i \geq 1$.

$$(2.2) \qquad \int_{-\infty}^{\infty} K_i(x) dx = \frac{1}{2} d_i^2 \; .$$

<u>Proof.</u> Since the proof of (2.2) is easy, we omit it here. Since

$\varphi(0) = \varphi'(0) = 0$, we have $\varphi'(x) = \int_{0}^{x} \varphi''(t) dt$ and $\varphi(x) = \int_{0}^{x} \varphi'(t) dt$.

It follows that φ' is an odd function and φ an even function.

Therefore

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$$(2.3) \qquad 0 \le \varphi(x) = \int_0^{|x|} \varphi'(t)dt = |x|\varphi'(|x|) - \int_0^{|x|} t\, \varphi''(t)dt$$

$$\le x\, \varphi'(x) .$$

The integrability of $\varphi(f_n)$ then follows from that of $f_n\varphi'(f_n)$. We also need the integrability of $d_i\varphi'(f_{i-1})$ for $i \ge 1$. Since $\varphi'' \ge 0$, φ' is nondecreasing. Therefore

$$|f_i|\varphi'(|f_{i-1}|)I(|f_{i-1}| \le |f_i|) \le |f_i|\varphi'(|f_i|)$$

and

$$|f_i|\varphi'(|f_{i-1}|)I(|f_{i-1}| > |f_i|) \le |f_{i-1}|\varphi'(|f_{i-1}|) .$$

Hence

$$|d_i\varphi'(f_{i-1})| = |d_i|\varphi'(|f_{i-1}|)|$$

$$\le |f_i|\varphi'(|f_{i-1}|) + |f_{i-1}|\varphi'(|f_{i-1}|)$$

$$\le |f_i|\varphi'(|f_i|) + 2|f_{i-1}|\varphi'(|f_{i-1}|) .$$

This implies the integrability of $d_i\varphi'(f_{i-1})$. We now derive (2.1). In the weak martingale case, the left hand side of (2.1) is equal to

$$(2.4) \qquad \sum_{i=1}^n E[\varphi(f_i) - \varphi(f_{i-1})]$$

$$= \sum_{i=1}^n E[\varphi(f_i) - \varphi(f_{i-1}) - d_i\varphi'(f_{i-1})]$$

$$= \sum_{i=1}^n E\left\{ \int_0^{d_i} \int_0^y \varphi''(f_{i-1}+x)dxdy \right\} I(d_i \ge 0)$$

$$+ \sum_{i=1}^n E\left\{ \int_{d_i}^0 \int_y^0 \varphi''(f_{i-1}+x)dxdy \right\} I(d_i < 0) .$$

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In the nonnegative weak submartingale case, $\varphi'(f_{i-1})$ is nonnegative and (2.4) holds with the first equality replaced by " $\geq$ ". Now $\varphi'' \geq 0$. So we may reverse the order of the doubly integration in (2.4). By this, the extreme right hand side of (2.4) yields

$$\sum_{i=1}^{n} E \int_{-\infty}^{\infty} \varphi''(f_{i-1}+x)K_i(x)dx$$

and the lemma is proved.

In the case where f is a martingale or a nonnegative submartingale, let τ be a stopping time. By replacing f in (2.1) by the stopped martingale or nonnegative submartingale f^τ, we obtain

$$(2.5) \qquad E\varphi(f_n^\tau) \geq \sum_{i=1}^{n} E \; I(\tau \geq i) \int_{-\infty}^{\infty} \varphi''(f_{i-1}+x)K_i(x)dx$$

where again equality holds in the martingale case. If the differences of f are mutually independent with zero means, $\varphi(x) = x^2$ and $\tau = \inf\{n: |f_n| > a\}$ where $a > 0$, then (2.5) immediately yields Kolmogorov's two inequalities in the proof of the three series theorem.

<u>Theorem 2.1.</u> Let ψ' be a nonnegative and even function which is nonincreasing on $[0,\infty)$, and let $\psi(x) = \int_0^x \psi'(t)dt$. Then

$$(2.6) \qquad ES^2(f)\psi'(f^*) \leq 2 \sup_n E|f_n|\psi(|f_n|) \; .$$

<u>Proof.</u> There is nothing to prove if the right hand side of (2.6) is infinite. So we assume it to be finite. Let $K_i(x)$ be as in Lemma 2.1. It is not difficult to see that, for $i \geq 1$, $f_{i-1}+x$ lies

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between f_{i-1} and f_i on $\{x: K_i(x) > 0\}$. Now let $\varphi'' = \psi'$,
$\varphi'(x) = \int_0^x \varphi''(t)dt = \psi(x)$ and $\varphi(x) = \int_0^x \varphi'(t)dt$. Then the integrability condition in Lemma 2.1 is satisfied and the lemma immediately yields

$$ES_n^2(f)\psi'(f_n^*) \leq 2E\varphi(f_n) \leq 2E|f_n|\psi(|f_n|) \leq 2 \sup_n E|f_n|\psi(|f_n|)$$

where the second inequality follows from (2.3). By letting $n \to \infty$ and applying Fatou's lemma, the theorem is proved.

We now deduce from (2.6) three special cases.

Corollary 2.1. We have

$$(2.7) \qquad E \frac{S^2(f)}{1+f^{*2}} \leq \pi \|f\|_1 \quad ;$$

$$(2.8) \qquad E \frac{S^2(f)}{1+f^*} \leq 2 \sup_n E|f_n|\log(1+|f_n|) \quad ;$$

$$(2.9) \qquad E \frac{S^2(f)}{f^{*2-p}} \leq \frac{2}{p-1} \|f\|_p^p, \; 1 < p \leq 2 \; .$$

Proof. For (2.7), let $\psi'(x) = (1+x^2)^{-1}$; and for (2.8), let $\psi'(x) = (1+|x|)^{-1}$. For (2.9), we first let $\psi'(x) = (a+|x|)^{p-2}$ where $a > 0$ and then let $a \downarrow 0$.

The inequality (2.7) generalizes and refines a result of Austin (1966) who proved that the square function of an L_1-bounded martingale is square integrable on any set where the maximal function is bounded above.

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3. Applications.

In this section, we use Corollary 2.1 to give easy proofs of Burkholder's $L \log L$ and L_p (for $1 < p \le 2$) inequalities for the square function of a martingale or a nonnegative submartingale. These inequalities were first proved by Burkholder (1966). Since then different proofs have been given. (See, for example, Gordon (1972), Burkholder (1973), Chao (1973) and Garsia (1973).)

Theorem 3.1. Let $f = (f_1, f_2, \ldots)$ be a martingale or a nonnegative submartingale. Then

$$(3.1) \qquad ES(f) \le 2 \left(\frac{e}{e-1} \right)^{1/2} \left[1 + \sup_n E|f_n| \log^+ |f_n| \right].$$

Proof. We shall use the following inequality which dates back to Young (1913). It can also be found in Doob (1953).

$$(3.2) \qquad a \log^+ b \le a \log^+ a + be^{-1} \quad \text{for} \quad a \ge 0 \quad \text{and} \quad b \ge 0.$$

Replacing f_i by $\lambda^{-1} f_i$ in (2.8) of Corollary 2.1 where $\lambda = Ef^*$, we obtain

$$(3.3) \qquad E \frac{S^2(f)}{\lambda + f^*} \le 2 \sup_n E|f_n| \log(1 + \lambda^{-1}|f_n|)$$

which by (3.2)

$$\le 2 \sup_n [E|f_n| \log^+|f_n| + (\lambda e)^{-1}(\lambda + E|f_n|)]$$

$$\le 2 [1 + \sup_n E|f_n| \log^+|f_n|].$$

Now applying the Cauchy-Schwarz inequality to

$$ES(f) = E\left(\frac{S^2(f)}{\lambda+f^*}\right)^{1/2}(\lambda+f^*)^{1/2}$$

and using (3.3) and the following inequality of Doob (1953) for sub-martingales,

$$\|f^*\|_1 \leq \left(\frac{e}{e-1}\right)[1 + \sup_n E|f_n|\log^+|f_n|] \ ,$$

we obtain (3.1). This proves the theorem.

<u>Theorem 3.2.</u> Let $f = (f_1, f_2, \ldots)$ be a martingale or a nonnegative submartingale. Then for $1 < p \leq 2$,

(3.4) $$\|S(f)\|_p \leq 2^{1/2}p^{1/2}q\|f\|_p$$

where $p^{-1} + q^{-1} = 1$.

<u>Proof.</u> Applying Hölder's inequality to

$$\|S(f)\|_p^p = E\left(\frac{S^2(f)}{f^{*2-p}}\right)^{\frac{1}{2}p}(f^{*p})^{1-\frac{1}{2}p} \ ,$$

we obtain

$$\|S(f)\|_p^p \leq \left(E\frac{S^2(f)}{f^{*2-p}}\right)^{\frac{1}{2}p}(\|f^*\|_p^p)^{1-\frac{1}{2}p}$$

which by (2.9) of Corollary 2.1

$$\leq \left(\frac{2}{p-1}\|f\|_p^p\right)^{\frac{1}{2}p}(\|f^*\|_p^p)^{1-\frac{1}{2}p} \ .$$

This together with the following inequality of Doob (1953) for sub-martingales,

$$\|f^*\|_p \leq q\|f\|_p \quad \text{for} \quad 1 < p < \infty \, , \; p^{-1} + q^{-1} = 1 \, ,$$

imply

$$\|S(f)\|_p \leq (\frac{2}{p-1})^{1/2} q^{1 - \frac{1}{2}p} \|f\|_p \leq 2^{1/2} p^{1/2} q\|f\|_p \, .$$

This proves the theorem.

The absolute constants in (3.1) and (3.4) seem to be the lowest ever obtained. In (3.4), the order of magnitude of the constant as $p \to 1$ is the same as that obtained by Burkholder (1973).

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